Steiner Quadruple Systems with Point-Regular Abelian Automorphism Groups

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Received: April 14th, 2010 *Revised:* December 20th, 2010

Abstract

In this paper we present a graph theoretic construction of Steiner quadruple systems (SQS) admitting abelian groups as point-regular automorphism groups. The resulting SQS has an extra property which we call A-reversibility, where A is the underlying abelian group. In particular, when A is a 2-group of exponent at most 4, it is shown that an A-reversible SQS always exists. When the Sylow 2-subgroup of A is cyclic, we give a necessary and sufficient condition for the existence of an A-reversible SQS, which is a generalization of a necessary and sufficient condition for the existence of a dihedral SQS by Piotrowski (1985). This enables one to construct A-reversible SQS for any abelian group A of order v such that for every prime divisor p of v there exists a dihedral $\mathrm{SQS}(2p)$.

AMS Subject Classification: 05E20, 05B05

Keywords: Steiner system; combinatorial design; graph; finite group.

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1 Introduction

Let t, k, λ and v be positive integers such that $t < k < v$. A t-(v, k, λ) *design* is an ordered pair $\mathcal{D} = (V, \mathcal{B})$ consisting of a set V of v points, and a collection $\mathcal{B}$ of k-subsets (called blocks) of V such that every t-subset of V occurs in exactly λ blocks. In particular, $\mathcal{D}$ is a *Steiner system* if $\lambda = 1$. A *Steiner quadruple system* (SQS) of order v, denoted by $\mathrm{SQS}(v)$, is a 3-$(v, 4, 1)$ design. A well known result by Hanani (1963) states that an $\mathrm{SQS}(v)$ exists if and only if $v \equiv 2$ or $4 \pmod 6$.

An *automorphism* of $\mathcal{D}$ is a permutation ξ on V such that $B^{\xi} \in \mathcal{B}$ for each $B \in \mathcal{B}$. The collection of all automorphisms of $\mathcal{D}$ forms a group, called the *full automorphism group*, and a subgroup of the full automorphism group is an *automorphism group* of $\mathcal{D}$. A finite group A acting on V induces a natural action on the set $\binom{V}{k}$ of all k-subsets of V. In a classical method of constructing t-designs one chooses suitable A-orbits of k-subsets and obviously the resulting design admits A as an automorphism group. It is well known that if A is t-transitive on V, then there exists a t-design with A as an automorphism group. Since there are few t-transitive groups with $t \geq 3$, we wish to develop a method of constructing t-designs, which works for permutation groups with low transitivity.

In this paper, we take an abelian group A of order v as the set of points, and construct $\mathrm{SQS}(v)$ whose set of blocks has some extra property. If A is an abelian group, then A acts on itself by translation. Let σ be the involutory automorphism of A defined by $a^{\sigma} = -a$. We regard A as a permutation group acting on A regularly, and form the semidirect product $\hat{A} = A \rtimes \langle \sigma \rangle$. The group $\hat{A}$ is a permutation group on A. A subset B of A is said to be *symmetric* if $B = -B + x$ for some $x \in A$, or equivalently, the orbit of B under $\hat{A}$ coincides with that of B under A. An $\mathrm{SQS}(v)$ on points A is said to be A-*reversible* if every block is symmetric and the set of all blocks is invariant under the action of $\hat{A}$. When A is the cyclic group $\mathbb{Z}_v$, then an A-reversible $\mathrm{SQS}(v)$ has been known as an S-cyclic $\mathrm{SQS}(v)$. It was Fitting (1915) who first proposed a graph theoretic construction of S-cyclic $\mathrm{SQS}(v)$. Later, his construction was taken up again and refined by Köhler (1979), in which the notion of *the first Köhler graph of order* v was introduced. In the same paper, among many other things, Köhler proved a fundamental theorem for S-cyclic $\mathrm{SQS}(v)$ such that the stabilizer of any quadruple under the action of $\mathbb{Z}_v$ equals the identity, which states that for $v \equiv 2$ or $10 \pmod{24}$, if the first Köhler graph of order v has a 1-factor, then there exists a S-cyclic $\mathrm{SQS}(v)$. After Köhler's work, some researchers have tried to construct S-cyclic $\mathrm{SQS}(v)$, on which some progress has been made, but far from settled in general. Without restriction of the stabilizers of quadruples, Piotrowski (1985)[Satz 14.1] proved a theorem stating that there exists a S-cyclic $\mathrm{SQS}(v)$ if and only if $v \equiv 0 \pmod 2$, $v \not\equiv 0 \pmod 3$, $v \not\equiv 0 \pmod 8$, $v \geq 4$ and for any prime divisor p of v there exists a S-cyclic $\mathrm{SQS}(2p)$. See Bitan and Etzion (1993); Feng et al. (2008); Siemon (1998, 1991, 1987) for more information on S-cyclic $\mathrm{SQS}(v)$.

The main purpose of this paper is to generalize Piotrowski's theorem on S-cyclic $\mathrm{SQS}(v)$ to A-reversible $\mathrm{SQS}(v)$. In Section 2, the concept of the *Köhler graph* is introduced as a generalization of first Köhler graphs of cyclic groups. In Section 3, the structure of Köhler graphs is investigated with special emphasis on degree of vertex and connected component. In Section 4, the number of orbits of certain triples and quadruples for an abelian group are counted. In Section 5, we assume that A is an abelian group of order $v \equiv 2$ or $4 \pmod 6$. We show that the special classes of triples and quadruples discussed in Section 4 could be incorporated into an A-reversible $\mathrm{SQS}(v)$, thereby reducing the existence of an A-reversible $\mathrm{SQS}(v)$ to a graph theoretic problem about the Köhler graph of A in Section 6. As an example, an infinite family of A-reversible $\mathrm{SQS}(2^n)$ is given for abelian groups $\mathbb{Z}_2^a \times \mathbb{Z}_4^b$. Finally in Section 7, we prove that for an abelian group A whose Sylow 2-subgroup is cyclic, the following statements are equivalent:

(i) There exists an A-reversible $\mathrm{SQS}(v)$;

(ii) $v \equiv 0 \pmod 2$, $v \not\equiv 0 \pmod 3$, $v \not\equiv 0 \pmod 8$, $v \geq 4$ and for any prime divisor p of v there exists a S-cyclic $\mathrm{SQS}(2p)$.

This is a generalization of a theorem of Piotrowski, reformulated by Siemon (1998)[p.93]. In Siemon (1991), it is shown that a S-cyclic $\mathrm{SQS}(2p)$ exists for any prime number $p \equiv 53$ or $77 \pmod{120}$ with $p < 500000$. Applying our theorem to this result shows that there exists an A-reversible $\mathrm{SQS}(v)$ for any abelian group A of order v which is twice a product of prime numbers p with $p \equiv 53$ or $77 \pmod{120}$ and $p < 500000$.

2 The Köhler graph of an abelian group

Throughout this section, we let A be an abelian group of order v. We regard A as a permutation group acting on A regularly, and form the semidirect product $\hat{A} = A \rtimes \langle\sigma\rangle$, where σ is the automorphism of A defined by $a^\sigma = -a$. The group $\hat{A}$ is a permutation group on A. For a subset X of A, let $\mathrm{Orb}_{\hat{A}}(X)$ denote the $\hat{A}$-orbit of X:

$$\mathrm{Orb}_{\hat{A}}(X) = \{X + a \mid a \in A\} \cup \{-X + a \mid a \in A\}.$$

For distinct nonzero elements $a_1, \dots, a_t \in A$, we abbreviate

$$\mathrm{Orb}_{\hat{A}}(\{0, a_1, \dots, a_t\})$$

as $[a_1, \dots, a_t]$. If $\{0, a, b\} \in \binom{A}{3}$, then

$$\begin{aligned} &\{X \mid \{0\} \cup X \in [a, b],\ 0 \notin X\} \\ &= \{\{a, b\}, \{-a, b - a\}, \{-b, a - b\}, \\ &\quad \{-a, -b\}, \{a, a - b\}, \{b, b - a\}\}. \end{aligned} \tag{1}$$

If $\{0, a, b, a + b\} \in \binom{A}{4}$, then

$$\begin{aligned} &\{Y \mid \{0\} \cup Y \in [a, b, a + b],\ 0 \notin Y\} \\ &= \{\{a, b, a + b\}, \{-a, b, -a + b\}, \\ &\quad \{a, -b, a - b\}, \{-a, -b, -a - b\}\}. \end{aligned} \tag{2}$$

Let

$$\begin{aligned} \mathcal{T} = \{[a, b] \mid a, b \in A,\ a \neq \pm b, \\ 2a \notin \{0, b, 2b\},\ 2b \notin \{0, a, 2a\}\}, \end{aligned} \tag{3}$$

$$\begin{aligned} \mathcal{E} = \{[a, b, a + b] \mid a, b \in A,\ 0 \notin \{2a, 2b\}, \\ \{\pm a, \pm 2a\} \cap \{\pm b, \pm 2b\} = \emptyset\}. \end{aligned} \tag{4}$$

Definition 2.1. The *Köhler graph* of A is the incidence structure $\mathcal{G} = (\mathcal{T}, \mathcal{E})$, where $\mathcal{T}, \mathcal{E}$ are defined in (3), (4), respectively, and $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}$ is incident with $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{E}$ if $B \supset T'$ for some $T' \in \mathrm{Orb}_{\hat{A}}(T)$.

Lemma 2.2. (i) *If* $\{0, a, b\} \in \binom{A}{3}$*, then* $[a, b] \in \mathcal{T}$ *if and only if*

$$a \neq \pm b, \quad 2a \notin \{0, b, 2b\} \text{ and } 2b \notin \{0, a, 2a\}. \tag{5}$$

(ii) *If* $\{0, a, b, a+b\} \in \binom{A}{4}$*, then* $[a, b, a+b] \in \mathcal{E}$ *if and only if*

$$0 \notin \{2a, 2b\} \text{ and } \{\pm a, \pm 2a\} \cap \{\pm b, \pm 2b\} = \emptyset. \tag{6}$$

Proof. (i) The "if" part is trivial. To prove the "only if" part, suppose that there exist $c, d \in A$ such that $[a, b] = [c, d]$ and

$$c \neq \pm d,\ 2c \notin \{0, d, 2d\},\ 2d \notin \{0, c, 2c\}. \tag{7}$$

By (1), we have $\{a, b\} \in \{\pm\{c, d\}, \pm\{c, c-d\}, \pm\{d, d-c\}\}$. Thus, either (5) holds, or $\{a, b\} \in \{\pm\{c, c-d\}, \pm\{d, d-c\}\}$ holds. In the latter case, replacing (c, d) by $(-c, -d)$, (d, c) or $(-d, -c)$ if necessary, we may assume $(a, b) = (c, c-d)$. Then by (7), we have $a \neq \pm(a-b),\ 2a \notin \{0, a-b, 2(a-b)\},\ 2(a-b) \notin \{0, a, 2a\}$, and (5) holds also in this case.

(ii) The "if" part is trivial. To prove the "only if" part, suppose that there exist $c, d \in A$ such that $[a, b, a+b] = [c, d, c+d]$ and

$$0 \notin \{2c, 2d\},\ \{\pm c, \pm 2c\} \cap \{\pm d, \pm 2d\} = \emptyset. \tag{8}$$

By (2), we have $\{a, b, a+b\} \in \{\pm\{c, d, c+d\}, \pm\{-c, d, -c+d\}\}$. Replacing (c, d) by $(-c, -d)$, $(-c, d)$ or $(c, -d)$ if necessary, we may assume $\{a, b, a+b\} = \{c, d, c+d\}$. This implies $\{a, b\} = \{c, d\}$, and (6) holds by (8). □

Lemma 2.3. *Suppose* $T \in \binom{B}{3}$ *and* $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{E}$*. Then* B *is the only member of* $\mathrm{Orb}_{\hat{A}}(B)$ *containing* T*.*

Proof. Without loss of generality we may assume $B = \{0, a, b, a+b\}$, with $a, b \in A$ satisfying (6). We first suppose $T = \{0, a, b\}$. Then

$$\begin{aligned}
&|\{B' \in \mathrm{Orb}_{\hat{A}}(B) \mid T \subset B'\}| \\
&= |\{B' \in [a, b, a+b] \mid T \subset B'\}| \\
&= |\{C \in \{\{a, b, a+b\}, \{-a, b-a, b\}, \{-b, a-b, a\}, \\
&\qquad \{-b, -a, -a-b\}\} \mid \{a, b\} \subset C\}| \\
&= |\{\{a, b, a+b\}\}| \\
&= 1.
\end{aligned}$$

Next suppose $T = \{a, b, a+b\}$. Then $T-(a+b) = \{0, -a, -b\}$ and $B-(a+b) = \{0, -a, -b, -a-b\}$. By the first case, we see that $B-(a+b)$ is the only member of $\mathrm{Orb}_{\hat{A}}(B)$ containing $T-(a+b)$. This gives the desired result.

Next suppose $T = \{0, a, a+b\}$ or $\{0, b, a+b\}$. Switching a and b if necessary, we may assume $T = \{0, a, a+b\}$. Then $T-a = \{0, -a, b\}$ and $B-a = \{0, -a, b, b-a\}$. By the first case, we see that $B-a$ is the only member of $\mathrm{Orb}_{\hat{A}}(B)$ containing $T-a$. This gives the desired result. □

Lemma 2.4. *Let* $[a,b] \in \mathcal{T}$. *Then*

(i) $[a,b] \notin \{[a,a+b],[a,b-a],[b,a-b]\}$,

(ii) $[a,b-a] \neq [b,a-b]$,

(iii) $[a,a+b]=[a,b-a]$ *if and only if* $(2a+b,5a)=(0,0)$.

(iv) $[a,a+b]=[b,a-b]$ *if and only if* $(a+2b,5b)=(0,0)$.

Proof. Since $[a,b] \in \mathcal{T}$, Lemma 2.2(i) implies that (5) holds. If $[a,b] \in \{[a,a+b],[a,b-a]\}$, then by (1), we have

$$\begin{aligned}&\{\{a,b\},\{-a,b-a\},\{-b,a-b\},\{-a,-b\},\{a,a-b\},\{b,b-a\}\}\\ &\cap\{\{a,a+b\},\{a,b-a\}\} \neq \emptyset.\end{aligned}$$

By (5), we have $a \notin \{b,-a,b-a,-b,a-b\}$, and so $\{a+b,b-a\} \cap \{b,a-b\} \neq \emptyset$, which is impossible by (5). Switching the role of a and b, we obtain $[a,b] \neq [b,a-b]$. This establishes (i).

Since $b \notin \{a,b-a,-a,b-2a,a-b,2a-b\}$ by (5), we have

$$\begin{aligned}\{b,a-b\} \notin \{&\{a,b-a\},\{-a,b-2a\},\{a-b,2a-b\}\\ &\{-a,a-b\},\{a,2a-b\},\{b-a,b-2a\}\}.\end{aligned}$$

Thus $[a,b-a] \neq [b,a-b]$ by (1). This proves (ii).

Also, we have

$$\begin{aligned}&[a,a+b]=[a,b-a]\\ &\iff \{a,a+b\} \in \{\{a,b-a\},\{-a,b-2a\},\\ &\qquad\qquad \{a-b,2a-b\},\{-a,a-b\},\\ &\qquad\qquad \{a,2a-b\},\{b-a,b-2a\}\} && \text{(by (1))}\\ &\iff (a,a+b)=(b-2a,-a) && \text{(by (5))}\\ &\iff b=3a=-2a,\end{aligned}$$

establishing (iii).

Finally, since $[a,a+b]=\mathrm{Orb}_{\hat{A}}(-\{0,a,a+b\}+(a+b))=[b,a+b]$, (iv) follows from (iii). □

The next lemma shows that the Köhler graph is indeed a graph, in the sense that every member of $\mathcal{E}$ is incident with exactly two members of $\mathcal{T}$. In Sawa (2010) the concept of Köhler graph is already introduced for an abelian group with cyclic Sylow 2-subgroup.

Lemma 2.5. *If* $[a,b,a+b] \in \mathcal{E}$, *then*

$$\{\mathrm{Orb}_{\hat{A}}(T) \mid T \subset \{0,a,b,a+b\},\ |T|=3\} = \{[a,b],[a,a+b]\}. \tag{9}$$

In particular, $[a,b,a+b]$ *is incident with exactly two distinct members* $[a,b]$, $[a,a+b]$ *of* $\mathcal{T}$.

Proof. Since $\{0,a,a+b\} = -\{0,b,a+b\} + (a+b)$, we have

$$[a,a+b] = [b,a+b]. \tag{10}$$

Since $\{0,a,b\} = -\{a,b,a+b\} + (a+b)$, we have

$$[a,b] = \mathrm{Orb}_{\hat{A}}(\{a,b,a+b\}). \tag{11}$$

Thus

$$\begin{aligned}
&\{\mathrm{Orb}_{\hat{A}}(T) \mid T \subset \{0,a,b,a+b\},\ |T| = 3\} \\
&= \{[a,b],[a,a+b],[b,a+b],\mathrm{Orb}_{\hat{A}}(\{a,b,a+b\})\} \\
&= \{[a,b],[a,a+b]\} \qquad\qquad \text{(by (10), (11))}.
\end{aligned}$$

Since $[a,b,a+b] \in \mathcal{E}$, Lemma 2.2 implies

$$a \neq \pm b,\ 2a \notin \{0,b,2b\},\ 2b \notin \{0,a,2a\} \tag{12}$$

$$a \neq \pm(a+b),\ 2a \notin \{0,a+b,2(a+b)\},\ 2(a+b) \notin \{0,a,2a\}. \tag{13}$$

By (12), (13), we have $[a,b] \in \mathcal{T}$, $[a,a+b] \in \mathcal{T}$, respectively. Therefore, the elements of $\mathcal{T}$ which are incident with $[a,b,a+b]$ are $[a,b]$ and $[a,a+b]$. By Lemma 2.4(i), we have $[a,b] \neq [a,a+b]$. $\square$

Finally, we show that the Köhler graph has no multiple edges. To do this, we first determine the edges incident with a given vertex.

Lemma 2.6. *If $[a,b] \in \mathcal{T}$, then the edges incident with $[a,b]$ are*

$$\{[a,b,a+b],[a,b,b-a],[a,b,a-b]\} \cap \mathcal{E}. \tag{14}$$

Proof. Observe that $[a,b]$ is incident with an orbit $\mathrm{Orb}_{\hat{A}}(B)$ if and only if $\{0,a,b\} \subset B'$ for some $B' \in \mathrm{Orb}_{\hat{A}}(B)$. In this case, as $0 \in B'$, we may assume $B' = \{0,c,d,c+d\}$ for some $c,d \in A$ by (2). Then

$$\{a,b\} \in \{\{c,d\},\{c,c+d\},\{d,c+d\}\},$$

hence

$$\{c,d,c+d\} \in \{\{a,b,a+b\},\{a,b,b-a\},\{a,b,a-b\}\}.$$

Thus

$$\mathrm{Orb}_{\hat{A}}(B) = \mathrm{Orb}_{\hat{A}}(B') \in \{[a,b,a+b],[a,b,b-a],[a,b,a-b]\}.$$

Conversely, every member of (14) is incident with $[a,b]$. $\square$

We conclude this section by a remark. The Köhler graph $\mathcal{G} = (\mathcal{T},\mathcal{E})$ is defined as an incidence structure, so it is nontrivial to prove that $\mathcal{G}$ has no multiple edges. Suppose that a vertex $[a,b] \in \mathcal{T}$ is incident with multiple edges. By Lemma 2.6, the possible edges incident with $[a,b]$ are

$$[a,b,a+b],[a,b,b-a],[a,b,a-b], \tag{15}$$

which are incident with the vertices

$$[a, a+b], [a, b-a], [b, a-b], \tag{16}$$

respectively, by Lemma 2.5. By Lemma 2.4(ii), we have $[b, a-b] \neq [a, b-a]$, hence the pair $[a, b, b-a], [a, b, a-b]$ does not form a pair of multiple edges sharing the common endpoints. Thus

$$[a, a+b] = [a, b-a] \text{ or } [b, a-b], \tag{17}$$

and $[a, b, a+b]$ is one of the multiple edges. In particular, $[a, b, a+b] \in \mathcal{E}$, and hence $2a+b \neq 0$ and $a+2b \neq 0$ by Lemma 2.2(ii). Then by Lemma 2.4(iii)–(iv), we have $[a, a+b] \neq [a, b-a], [b, a-b]$, contradicting (17). Therefore, the Köhler graph $\mathcal{G}$ has no multiple edges.

3 The structure of the Köhler graphs

Lemma 3.1. *Let* $[a, b] \in \mathcal{T}$.

(i) $[a, a+b] \in \mathcal{T}$ *if and only if* $0 \notin \{2a+b, a+2b, 2a+2b\}$,

(ii) $[a, b-a] \in \mathcal{T}$ *if and only if* $0 \notin \{3a-b, 3a-2b, 4a-2b\}$.

Proof. (i)

$$\begin{aligned}
&[a, a+b] \in \mathcal{T} \\
&\iff a \neq \pm(a+b),\ 2a \notin \{0, a+b, 2(a+b)\}, \\
&\qquad 2(a+b) \notin \{0, a, 2a\} && \text{(by Lemma 2.2(i))} \\
&\iff 0 \notin \{2a+b, a+2b, 2a+2b\} && \text{(by (5)).}
\end{aligned}$$

(ii)

$$\begin{aligned}
&[a, b-a] \in \mathcal{T} \\
&\iff a \neq \pm(b-a),\ 2a \notin \{0, b-a, 2(b-a)\}, \\
&\qquad 2(b-a) \notin \{0, a, 2a\} && \text{(by Lemma 2.2(i))} \\
&\iff 0 \notin \{3a-b, 3a-2b, 4a-2b\} && \text{(by (5)).}
\end{aligned}$$

□

For $v \in \mathcal{T}$ we denote the set of neighbors of v by $N(v)$.

Lemma 3.2. *Let* $[a, b] \in \mathcal{T}$. *Then*

$$N([a, b]) = \{[a, a+b], [b, a-b], [a, b-a]\} \cap \mathcal{T}. \tag{18}$$

Proof. By Lemmas 2.5 and 2.6, $N([a, b])$ is contained in (18).

Conversely,

$$\begin{aligned}
&[a, a+b] \in \mathcal{T} \\
&\iff 0 \notin \{2a+b, a+2b, 2a+2b\} && \text{(by Lemma 3.1(i))} \\
&\iff [a, b, a+b] \in \mathcal{E} && \text{(by Lemma 2.2(ii))} \\
&\implies [a, a+b] \in N([a, b]) && \text{(by Lemma 2.5),}
\end{aligned}$$

and

$$\begin{aligned}
&[a, b-a] \in \mathcal{T} \\
&\iff 0 \notin \{3a-b, 3a-2b, 4a-2b\} && \text{(by Lemma 3.1(i))} \\
&\iff [a, b, b-a] \in \mathcal{E} && \text{(by Lemma 2.2(ii)).} \\
&\implies [a, b-a] \in N([a, b]) && \text{(by Lemma 2.5).}
\end{aligned}$$

Finally, switching a and b, we see that $[b, a-b] \in \mathcal{T}$ implies $[b, a-b] \in N([a, b])$. $\square$

Lemma 3.3. *Let* $[a, b] \in \mathcal{T}$. *Then the degree of* $[a, b]$ *is* 3 *if and only if*

$$0 \notin \{2a+b, a+2b, 2a+2b, 3a-b, 3a-2b, 4a-2b, 3b-a, 3b-2a, 4b-2a\}.$$

Proof. The degree of $[a, b]$ is at most 3 by Lemma 3.2. The degree is exactly 3 if and only if $[a, a+b], [a, b-a]$ and $[b, a-b]$ are distinct elements of $\mathcal{T}$. By Lemma 2.4(ii)–(iv), $[a, a+b], [a, b-a]$ and $[b, a-b]$ are distinct if and only if $(2a+b, 5a) \neq (0, 0)$ and $(a+2b, 5b) \neq (0, 0)$. Therefore,

$$\begin{aligned}
&\text{degree of } [a, b] \text{ is not } 3 \\
&\iff (2a+b, 5a) = (0,0) \text{ or } (a+2b, 5b) = (0,0) \text{ or} \\
&\qquad \{[a, a+b], [a, b-a], [b, a-b]\} \not\subset \mathcal{T} \\
&\iff (2a+b, 5a) = (0,0) \text{ or } (a+2b, 5b) = (0,0) \text{ or} \\
&\qquad 0 \in \{2a+b, a+2b, 2a+2b\} \text{ or} \\
&\qquad 0 \in \{3a-b, 3a-2b, 4a-2b\} \text{ or} \\
&\qquad 0 \in \{3b-a, 3b-2a, 4b-2a\} && \text{(by Lemma 3.1)} \\
&\iff 0 \in \{2a+b, a+2b, 2a+2b\} \\
&\qquad\quad \cup \{3a-b, 3a-2b, 4a-2b\} \\
&\qquad\quad \cup \{3b-a, 3b-2a, 4b-2a\}.
\end{aligned}$$

$\square$

Example 3.4. The Köhler graph of $\mathbb{Z}_4^2$ is the 3-cube. Indeed, let $A = \langle g_1 \rangle \oplus \langle g_2 \rangle \simeq \mathbb{Z}_4^2$. The Köhler graph of A has 8 vertices

$$v_i = \begin{cases} [g_1, (i-1)g_1 + g_2] & \text{if } i = 1, 2, 3, 4; \\ [g_1 + 2g_2, (i-4)g_1 + (2i-7)g_2] & \text{if } i = 5, 6, 7, 8. \end{cases} \tag{19}$$

where the neighbors $N(v_i)$ of v_i are determined as follows:

$$N(v_1) = \{v_2, v_4, v_5\},\ N(v_2) = \{v_1, v_3, v_8\},$$
$$N(v_3) = \{v_2, v_4, v_7\},\ N(v_4) = \{v_1, v_3, v_6\},$$
$$N(v_5) = \{v_1, v_8, v_6\},\ N(v_6) = \{v_4, v_7, v_5\},$$
$$N(v_7) = \{v_3, v_8, v_6\},\ N(v_8) = \{v_2, v_7, v_5\}.$$

Lemma 3.5. *Let $[a, b] \in \mathcal{T}$. Then $[a, b]$ is an isolated vertex of $\mathcal{G}$ if and only if one of the following conditions holds:*

(i) $[a, b] = [a, 3a]$ *with* $0 \in \{7a, 8a\}$*, or* $[a, b] = [3b, b]$ *with* $0 \in \{7b, 8b\}$*;*

(ii) $[a, b] = [a, -a + h]$ *with* $6a = 0$ *and* $2h = 0$.

Proof. By Lemma 3.2, $[a, b]$ is an isolated vertex of $\mathcal{G}$ if and only if

$$[a, a + b] \notin \mathcal{T},\ [b, a - b] \notin \mathcal{T},\ [a, b - a] \notin \mathcal{T}.$$

By Lemma 3.1, this occurs precisely when

$$\begin{aligned} 0 \in &\{2a + b, a + 2b, 2a + 2b\} \cap \{3a - b, 3a - 2b, 4a - 2b\} \\ &\cap \{3b - a, 3b - 2a, 4b - 2a\}. \end{aligned}$$

A tedious computation establishes the desired result. □

Example 3.6. Let $A = \langle a \rangle$ be the cyclic group of order 7 or 8. Then by Lemma 3.5, the Köhler graphs of $\mathbb{Z}_7$, $\mathbb{Z}_8$ both consist of a single vertex $[a, 3a]$.

Lemma 3.7. *Suppose that $[a, b] \in \mathcal{T}$ and $[c, d] \in \mathcal{T}$ belong to the same connected component of $\mathcal{G}$. Then $\langle a, b \rangle = \langle c, d \rangle$.*

Proof. It suffices to prove the assertion when $[c, d] = [a, b]$ or $[c, d]$ is adjacent to $[a, b]$. In the former case, the result follows immediately from (1). In the latter case, the result follows from Lemma 3.2. □

Lemma 3.8. *Let A' be a subgroup of A. Then the Köhler graph of A' is isomorphic to a union of connected components of $\mathcal{G}$. In particular, every connected component of the Köhler graph of A is isomorphic to a connected component of the Köhler graph of a subgroup of A generated by two elements.*

Proof. For distinct nonzero elements $a_1, \ldots, a_t \in A'$, we abbreviate

$$\mathrm{Orb}_{\hat{A}'}(\{0, a_1, \ldots, a_t\}) = [a_1, \ldots, a_t]'.$$

Let $\mathcal{T}(A')$, $\mathcal{E}(A')$ denote the sets defined by (3), (4), respectively, for the group A'. By (1) and Lemma 2.2(i), we have an injective mapping $\phi : \mathcal{T}(A') \to \mathcal{T}$ defined by $\phi([a, b]') = [a, b]$. We claim that ϕ is an isomorphism from the Köhler graph of A' onto

a union of some connected components of $\mathcal{G}$. Indeed, for $[a,b]' \in \mathcal{T}(A')$, Lemma 3.2 implies that the neighbors of $[a,b]'$ are

$$\{[a,a+b]', [a,b-a]', [b,a-b]'\} \cap \mathcal{T}(A'). \tag{20}$$

The mapping ϕ sends (20) to (18) which is the set of neighbors of $[a,b]$ in $\mathcal{G}$ by Lemma 3.2. This proves the claim.

Let $\mathcal{C} \subset \mathcal{T}$ be a connected component of $\mathcal{G}$, and let $[a,b] \in \mathcal{C}$. Take A' to be the subgroup of A generated by a, b. Then Lemma 3.7 implies $\mathcal{C} \subset \phi(\mathcal{T}(A'))$. Since ϕ is an isomorphism, $\mathcal{C}$ is isomorphic to a connected component of the Köhler graph of A'. □

Lemma 3.9. *Suppose $a, b \in A$ satisfy*

$$2a \notin \langle b\rangle \textit{ and } 2b \notin \langle a\rangle. \tag{21}$$

Then there exists a cycle containing the edge $[a,b,a+b]$ in the Köhler graph of A.

Proof. Let

$$\mathcal{C} = \{[a, ta+b, (t+1)a+b] \mid t \in \mathbb{Z}\}.$$

We claim that for any integer t,

$$[a, ta+b, (t+1)a+b] \in \mathcal{E}. \tag{22}$$

By (4), this is equivalent to

$$0 \notin \{2a, 2(ta+b)\} \text{ and } \{a, 2a\} \cap \{\pm(ta+b), \pm 2(ta+b)\} = \emptyset,$$

whose validity follows from the assumption (21). This establishes (22).

Observe that, for $s, t \in \mathbb{Z}$, we have

$$\begin{aligned} &[a, ta+b] = [a, sa+b] \\ &\iff \{a, ta+b\} \in \{\{a, sa+b\}, \{-a, (s-1)a+b\}, \\ &\qquad\qquad \{-sa-b, (1-s)a-b\}, \{-a, -sa-b\}, \\ &\qquad\qquad \{a, (1-s)a-b\}, \{sa+b, (s-1)a+b\}\} \qquad \text{(by (1))} \\ &\iff ta = sa \qquad\qquad \text{(by (21)).} \end{aligned}$$

Hence, by Lemma 2.5,

$$\{[a, ta+b] \mid 0 \le t < |\langle a\rangle|\}$$

is a set of $|\langle a\rangle|$ distinct vertices of $\mathcal{G}$ which form a cycle of length $|\langle a\rangle| \ge 3$. □

4 Special orbits of triples and quadruples

Throughout this section, we let A be an abelian group of order $v \equiv 2$ or $4 \pmod 6$. Let

$$\begin{aligned} \Omega_1(A) &= \{a \in A \mid 2a = 0\}, & \omega_1 &= |\Omega_1(A)|, \\ \Omega_2(A) &= \{a \in A \mid 4a = 0\}, & \omega_2 &= |\Omega_2(A)|. \end{aligned}$$

Let $\left[\begin{smallmatrix}\Omega_1(A)\\2\end{smallmatrix}\right]$ denote the set of all subgroups of order 4 in $\Omega_1(A)$. Since $\Omega_1(A)$ is an elementary abelian 2-group, $\Omega_1(A)$ can be regarded as a vector space over the finite field $\mathbb{F}_2$ of two elements. Then $\left[\begin{smallmatrix}\Omega_1(A)\\2\end{smallmatrix}\right]$ is just the set of 2-dimensional subspaces of $\Omega_1(A)$, and by Dembowski (1968)[p.28],

$$\left|\begin{bmatrix}\Omega_1(A)\\2\end{bmatrix}\right| = \frac{(\omega_1-1)(\omega_1-2)}{6}. \tag{23}$$

Let

$$\mathcal{T}_1 = \{[a,-a] \mid a \in A \setminus \Omega_1(A)\}, \tag{24}$$
$$\mathcal{T}_2 = \{[a,h] \mid a \in A \setminus \{0\},\ h \in \Omega_1(A) \setminus \{0,a\}\}. \tag{25}$$

By (1), if $b = -a \neq a$, then

$$\{X \mid \{0\} \cup X \in [a,-a],\ 0 \notin X\} = \{\{a,-a\},\{a,2a\},\{-a,-2a\}\}, \tag{26}$$

and if $a \neq 0$ and $h \in \Omega_1(A) \setminus \{0,a\}$, then

$$\{X \mid \{0\} \cup X \in [a,h],\ 0 \notin X\} = \{\pm\{a,h\},\pm\{a,h+a\},\pm\{h,h+a\}\}. \tag{27}$$

Fix an element h_0 of order 2 in A. Set

$$\mathcal{Q}_1 = \{[a,-a,h_0] \mid a \in A \setminus \Omega_1(A)\}, \tag{28}$$
$$\mathcal{Q}_2 = \{[a,h,h+a] \mid a \in A \setminus \Omega_1(A)\ , h \in \Omega_1(A) \setminus \langle h_0\rangle,\ 2a \neq h\}, \tag{29}$$
$$\mathcal{Q}_3 = \{[h,h',h+h'] \mid h,h' \in \Omega_1(A) \setminus \{0\},\ h \neq h'\}. \tag{30}$$

If $\{0,a,-a,h\} \in \binom{A}{4}$ and $h \in \Omega_1(A)$, then

$$\begin{aligned}&\{Y \mid \{0\} \cup Y \in [a,-a,h],\ 0 \notin Y\}\\ &= \{\{a,-a,h\},\{h+a,h-a,h\},\\ &\qquad \{a,2a,h+a\},\{-a,-2a,h-a\}\}.\end{aligned} \tag{31}$$

Lemma 4.1. *We have $\binom{A}{3}/\hat{A} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}$, and $\mathcal{T} \cap (\mathcal{T}_1 \cup \mathcal{T}_2) = \emptyset$, where $\binom{A}{3}/\hat{A}$ is the orbit decomposition of $\binom{A}{3}$ under $\hat{A}$.*

Proof. Let $a,b \in A \setminus \{0\}$ be distinct. Then by (26),

$$\begin{aligned}[a,b] \in \mathcal{T}_1 &\iff \exists c \in A \text{ s.t. } \{a,b\} \in \{\{c,-c\},\{c,2c\},\{-c,-2c\}\}\\ &\iff a=-b,\ a=2b \text{ or } 2a=b,\end{aligned}$$

and by (27),

$$\begin{aligned}[a,b] \in \mathcal{T}_2 \iff &\ \exists c \in A,\ h \in \Omega_1(A) \text{ s.t.}\\ &\quad \{a,b\} \in \{\pm\{c,h\},\pm\{h,h+c\},\pm\{c,h+c\}\}\\ \iff &\ 2a=0,\ 2b=0 \text{ or } 2a=2b.\end{aligned}$$

Combining these we conclude $[a,b] \notin \mathcal{T}_1 \cup \mathcal{T}_2$ if and only if $[a,b] \in \mathcal{T}$. ☐

Lemma 4.2. *Let* $a, b \in A \setminus \Omega_1(A)$. *Then*

(i) $[a, -a] = [b, -b]$ *if and only if* $a = \pm b$,

(ii) $[a, -a, h_0] = [b, -b, h_0]$ *if and only if* $b \in \{\pm a, h_0 \pm a\}$.

Proof. (i) By (26),

$$\begin{aligned} [a, -a] = [b, -b] &\iff \{b, -b\} \in \{\{a, -a\}, \{a, 2a\}, \{-a, -2a\}\} \\ &\iff a = \pm b, \end{aligned}$$

since $v \not\equiv 0 \pmod 3$.

(ii) By (31),

$$\begin{aligned} &[a, -a, h_0] = [b, -b, h_0] \\ &\iff \{b, -b, h_0\} \in \{\{a, -a, h_0\}, \{h_0 + a, h_0 - a, h_0\}, \\ &\qquad\qquad \{a, 2a, h_0 + a\}, \{-a, -2a, h_0 - a\}\} \\ &\iff b \in \{\pm a, h_0 \pm a\} \\ &\qquad \text{or } h_0 = 2a \text{ and } \{b, -b\} \in \{\{a, 3a\}, \{-a, a\}\} \\ &\iff b \in \{\pm a, h_0 \pm a\}. \end{aligned}$$

$\square$

Lemma 4.3. *Let* $a \in A \setminus \Omega_1(A)$, $b \in A \setminus \{0\}$ *and* $h \in \Omega_1(A) \setminus \{0, b\}$. *Then*

(i) $[a, -a] = [b, h]$ *if and only if* $a = \pm b$ *and* $2a = h$. *In particular,* $[a, -a] \in \mathcal{T}_2$ *if and only if* $a \in \Omega_2(A)$.

(ii) $[a, -a, h_0] = [b, h, h + b]$ *if and only if* $a = \pm b$ *and* $2a = h_0 = h$.

Proof. (i) By (26),

$$\begin{aligned} [a, -a] = [b, h] &\iff \{b, h\} \in \{\{a, -a\}, \{a, 2a\}, \{-a, -2a\}\} \\ &\iff (b, h) \in \{(a, 2a), (-a, -2a)\} \\ &\iff a = \pm b \text{ and } 2a = h. \end{aligned}$$

Thus

$$\begin{aligned} [a, -a] \in \mathcal{T}_2 &\iff \exists b \in A \setminus \{0\},\ \exists h \in \Omega_1(A) \setminus \{0, b\} \\ &\qquad \text{s.t. } a = \pm b \text{ and } h = 2a \\ &\iff a \in \Omega_2(A) \text{ and } \exists b \in \{\pm a\} \setminus \{0, 2a\} \\ &\iff a \in \Omega_2(A), \end{aligned}$$

since $\{\pm a\} \cap \{0, 2a\} = \emptyset$ when $a \in \Omega_2(A) \setminus \Omega_1(A)$.

(ii) By (31),

$$\begin{aligned}
&[a,-a,h_0]=[b,h,h+b]\\
&\iff \{b,h,h+b\}\in\{\{a,-a,h_0\},\{h_0+a,h_0-a,h_0\},\\
&\qquad\qquad\qquad\qquad \{a,2a,h_0+a\},\{-a,-2a,h_0-a\}\}\\
&\iff h_0=h \text{ and } \{b,h_0+b\}\in\{\{a,-a\},\{h_0+a,h_0-a\}\}\\
&\qquad \text{or } 2a=h \text{ and } \{b,h+b\}\in\{\pm\{a,h_0+a\}\}\\
&\iff h_0=h,\ b\in\{\pm a,h_0\pm a\} \text{ and } 2a=h_0,\\
&\qquad \text{or } 2a=h,\ b\in\{\pm a,h_0\pm a\} \text{ and } h_0=h\\
&\iff a=\pm b \text{ and } 2a=h_0=h.
\end{aligned}$$

□

Lemma 4.4. *Let $a,a'\in A\setminus\{0\}$, $h\in\Omega_1(A)\setminus\{0,a\}$, and $h'\in\Omega_1(A)\setminus\{0,a'\}$. Then the following are equivalent:*

(i) $[a,h]=[a',h']$,

(ii) $[a,h,h+a]=[a',h',h'+a']$,

(iii)

$$a,a'\in\Omega_1(A) \textit{ and } \langle a,h\rangle=\langle a',h'\rangle, \tag{32}$$

or

$$a,a'\notin\Omega_1(A),\ h=h' \textit{ and } a'\in\{\pm a,h\pm a\}. \tag{33}$$

Proof. Suppose $a\in\Omega_1(A)$. Then $[a,h]=[a',h']$ or $[a,h,h+a]=[a',h',h'+a']$ implies $a'\in\Omega_1(A)$ by (27) or (2). If $a,a'\in\Omega_1(A)$, then

$$\begin{aligned}
&[a,h]=[a',h']\\
&\iff \{a',h'\}\in\{\{a,h\},\{h,h+a\},\{a,h+a\}\} &&\text{(by (27))}\\
&\iff \langle a,h\rangle=\langle a',h'\rangle\\
&\iff \{a,h,h+a\}=\{a',h',h'+a'\}\\
&\iff [a,h,h+a]=[a',h',h'+a'] &&\text{(by (2)).}
\end{aligned}$$

Now suppose $a,a'\notin\Omega_1(A)$. Then

$$\begin{aligned}
&[a,h]=[a',h']\\
&\iff \{a',h'\}\in\{\pm\{a,h\},\pm\{h,h+a\},\pm\{a,h+a\}\} &&\text{(by (27))}\\
&\iff \{a',h'\}\in\{\pm\{a,h\},\pm\{h,h+a\}\}\\
&\iff h=h' \text{ and } a'\in\{\pm a,h\pm a\}\\
&\iff h=h' \text{ and } \{a',h'+a'\}=\pm\{a,h+a\}\\
&\iff \{a',h',h'+a'\}=\pm\{a,h,h+a\}\\
&\iff [a,h,h+a]=[a',h',h'+a'] &&\text{(by (2)).}
\end{aligned}$$

□

Lemma 4.5. *Let $a \in A \setminus \Omega_1(A)$, $T \in [a, -a]$ and $B \in [a, -a, h_0]$. Then*

$$|\{a' \in A \setminus \Omega_2(A) \mid T \in [a', -a']\}| = 2 \quad \text{if } a \notin \Omega_2(A), \tag{34}$$
$$|\{a' \in A \setminus \Omega_1(A) \mid 2a' = h_0,\ B \in [a', -a', h_0]\}| = 2 \quad \text{if } 2a = h_0, \tag{35}$$
$$|\{a' \in A \setminus \Omega_1(A) \mid 2a' \neq h_0,\ B \in [a', -a', h_0]\}| = 4 \quad \text{if } 2a \neq h_0. \tag{36}$$

Proof. If $a \notin \Omega_2(A)$, then by Lemma 4.2(i), we have

$$\begin{aligned}\{a' \in A \setminus \Omega_2(A) \mid T \in [a', -a']\} &= \{a' \in A \setminus \Omega_2(A) \mid [a, -a] = [a', -a']\} \\ &= \{\pm a\}.\end{aligned}$$

Thus

$$|\{a' \in A \setminus \Omega_2(A) \mid T \in [a', -a']\}| = |\{\pm a\}| = 2.$$

Suppose $a \in A \setminus \Omega_1(A)$. By Lemma 4.2(ii), we have

$$\begin{aligned}&\{a' \in A \setminus \Omega_1(A) \mid B \in [a', -a', h_0]\} \\ &= \{a' \in A \setminus \Omega_1(A) \mid [a, -a, h_0] = [a', -a', h_0]\} \\ &= \{\pm a, h_0 \pm a\} \\ &\subset \begin{cases}\{a' \in A \setminus \Omega_1(A) \mid 2a' = h_0\} & \text{if } 2a = h_0, \\ \{a' \in A \setminus \Omega_1(A) \mid 2a' \neq h_0\} & \text{otherwise.}\end{cases}\end{aligned}$$

Thus

$$\{a' \in A \setminus \Omega_1(A) \mid 2a' = h_0,\ B \in [a', -a', h_0]\} = |\{\pm a\}| = 2$$

if $2a = h_0$, and

$$\{a' \in A \setminus \Omega_1(A) \mid 2a' \neq h_0,\ B \in [a', -a', h_0]\} = |\{\pm a, h_0 \pm a\}| = 4$$

if $2a \neq h_0$. □

Lemma 4.6. *Let $a \in A \setminus \Omega_1(A)$, $h \in \Omega_1(A) \setminus \{0\}$, $T \in [a, h]$ and $B \in [a, h, h + a]$. Then*

$$|\{a' \in A \setminus \Omega_1(A) \mid 2a' = h,\ T \in [a', h]\}| = 2 \quad \text{if } 2a = h, \tag{37}$$
$$|\{a' \in A \setminus \Omega_1(A) \mid 2a' \neq h,\ T \in [a', h]\}| = 4 \quad \text{if } 2a \neq h, \tag{38}$$
$$|\{a' \in A \setminus \Omega_1(A) \mid 2a' = h,\ B \in [a', h, h + a']\}| = 2 \quad \text{if } 2a = h, \tag{39}$$
$$|\{a' \in A \setminus \Omega_1(A) \mid 2a' \neq h,\ B \in [a', h, h + a']\}| = 4 \quad \text{if } 2a \neq h. \tag{40}$$

Proof. By Lemma 4.4, we have

$$\begin{aligned}&\{a' \in A \setminus \Omega_1(A) \mid T \in [a', h]\} \\ &= \{a' \in A \setminus \Omega_1(A) \mid [a, h] = [a', h]\} \\ &= \{\pm a, h \pm a\} \\ &\subset \begin{cases}\{a' \in A \setminus \Omega_1(A) \mid 2a' = h\} & \text{if } 2a = h, \\ \{a' \in A \setminus \Omega_1(A) \mid 2a' \neq h\} & \text{otherwise.}\end{cases}\end{aligned}$$

Thus

$$|\{a' \in A \setminus \Omega_1(A) \mid 2a' = h,\ T \in [a', h]\}| = |\{\pm a\}| = 2$$

if $2a = h$, and

$$|\{a' \in A \setminus \Omega_1(A) \mid 2a' \neq h,\ T \in [a', h]\}| = |\{\pm a, h \pm a\}| = 4$$

if $2a \neq h$.

Also, by Lemma 4.4, we have

$$\begin{aligned}
&\{a' \in A \setminus \Omega_1(A) \mid B \in [a', h, h + a']\} \\
&= \{a' \in A \setminus \Omega_1(A) \mid [a, h, h + a] = [a', h, h + a']\} \\
&= \{\pm a, h \pm a\} \\
&\subset \begin{cases} \{a' \in A \setminus \Omega_1(A) \mid 2a' = h\} & \text{if } 2a = h, \\ \{a' \in A \setminus \Omega_1(A) \mid 2a' \neq h\} & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus

$$|\{a' \in A \setminus \Omega_1(A) \mid 2a' = h,\ B \in [a', h, h + a']\}| = |\{\pm a\}| = 2$$

if $2a = h$, and

$$|\{a' \in A \setminus \Omega_1(A) \mid 2a' \neq h,\ B \in [a', h, h + a']\}| = |\{\pm a, h \pm a\}| = 4$$

if $2a \neq h$. □

Lemma 4.7. (i) *Let $a \in A \setminus \Omega_1(A)$. Then*

$$|[a, -a]| = v, \tag{41}$$

and

$$|[a, -a, h_0]| = \begin{cases} \frac{v}{4} & \text{if } 2a = h_0, \\ v & \text{otherwise.} \end{cases} \tag{42}$$

(ii) *Let $a \in A \setminus \{0\}$, $h \in \Omega_1(A) \setminus \{0\}$, and $a \neq h$. Then*

$$|[a, h]| = \begin{cases} v & \text{if } 2a = h \text{ or } a \in \Omega_1(A), \\ 2v & \text{otherwise,} \end{cases} \tag{43}$$

and

$$|[a, h, h + a]| = \begin{cases} \frac{v}{2} & \text{if } a \notin \Omega_1(A) \text{ and } 2a \neq h, \\ \frac{v}{4} & \text{if } a \in \Omega_1(A). \end{cases} \tag{44}$$

Proof. If $a_1, \dots, a_t \in A \setminus \{0\}$ are distinct, then counting the number of pairs (x, T) with $x \in T \in [a_1, \dots, a_t]$, we find

$$\begin{aligned}
(t+1)|[a_1, \dots, a_t]| &= \sum_{x \in A} |\{T \in [a_1, \dots, a_t] \mid x \in T\}| \\
&= \sum_{x \in A} |\{T \in [a_1, \dots, a_t] \mid 0 \in T\}| \\
&= v|\{T \in \binom{A}{t} \mid \{0\} \cup T \in [a_1, \dots, a_t]\}|. \qquad (45)
\end{aligned}$$

Thus, for $a \in A \setminus \Omega_1(A)$, (26) implies

$$|[a, -a]| = \frac{v}{3}|\{\{a, -a\}, \{a, 2a\}, \{-a, -2a\}\}|, \qquad (46)$$

and (31) implies

$$\begin{aligned}
|[a, -a, h_0]| = \frac{v}{4}|\{&\{a, -a, h_0\}, \{h_0, h_0 + a, h_0 - a\}, \\
&\{a, 2a, h_0 + a\}, \{-a, -2a, h_0 - a\}\}|. \qquad (47)
\end{aligned}$$

Since $v \not\equiv 0 \pmod 3$, we have $3a \neq 0$ for any $a \in A \setminus \{0\}$. Thus (41) follows from (46), while (42) follows from (47).

As for (ii), if $a \in A \setminus \{0\}$ and $h \in \Omega_1(A) \setminus \{0\}$ and $a \neq h$, then (27) and (45) imply

$$|[a, h]| = \frac{v}{3}|\{\pm\{a, h\}, \pm\{h, h + a\}, \pm\{a, h + a\}\}|, \qquad (48)$$

while (2) and (45) imply

$$|[a, h, h + a]| = \frac{v}{4}|\{\{a, h, h + a\}, \{-a, h, h - a\}\}|. \qquad (49)$$

By (48), we have

$$\begin{aligned}
|[a, h]| &= \begin{cases} \frac{v}{3}|\{\pm\{a, 2a\}, \pm\{2a, 3a\}, \pm\{a, 3a\}\}| & \text{if } 2a = h, \\ \frac{v}{3}|\{\{a, h\}, \{h, h + a\}, \{a, h + a\}\}| & \text{if } a \in \Omega_1(A), \\ \frac{v}{3}|\{\pm\{a, h\}, \pm\{h, h + a\}, \pm\{a, h + a\}\}| & \text{otherwise}, \end{cases} \\
&= \begin{cases} \frac{v}{3}|\{\{a, 2a\}, \{-a, -2a\}, \{a, -a\}\}| & \text{if } 2a = h, \\ v & \text{if } a \in \Omega_1(A), \\ 2v & \text{otherwise}, \end{cases} \\
&= \begin{cases} v & \text{if } 2a = h \text{ or } a \in \Omega_1(A), \\ 2v & \text{otherwise}. \end{cases}
\end{aligned}$$

This proves (43). Finally, (44) follows from (49). $\square$

We now compute the number of triples T satisfying $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1 \cup \mathcal{T}_2$.

Lemma 4.8. *We have*

$$|\{T \mid \mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1 \cup \mathcal{T}_2\}| = \frac{1}{2}v^2\omega_1 - \frac{1}{6}v(2\omega_1^2 + 3\omega_2 - 2).$$

Proof. Counting the number of pairs (a,T) with $a \in A \setminus \Omega_1(A)$, $h \in \Omega_1(A) \setminus \{0\}$, $2a = h$ and $T \in [a,h]$ using (37), we find

$$2\left|\bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ 2a=h}} [a,h]\right| = \sum_{\substack{a\in A\setminus\Omega_1(A)\\ 2a=h}} |[a,h]|.$$

Thus, by (43), we have

$$\left|\bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ 2a=h}} [a,h]\right| = \frac{v}{2}|\{a \in A \setminus \Omega_1(A) \mid 2a = h\}|. \tag{50}$$

Counting the number of pairs (a,T) with $a \in A \setminus \Omega_1(A)$, $2a \neq h$ and $T \in [a,h]$ using (38), we find

$$4\left|\bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ 2a\neq h}} [a,h]\right| = \sum_{\substack{a\in A\setminus\Omega_1(A)\\ 2a\neq h}} |[a,h]|.$$

Thus, by (43), we have

$$\left|\bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ 2a\neq h}} [a,h]\right| = \frac{v}{2}|\{a \in A \setminus \Omega_1(A) \mid 2a \neq h\}|. \tag{51}$$

Therefore,

$$\begin{aligned}
&\left|\bigcup_{h\in\Omega_1(A)\setminus\{0\}} \bigcup_{a\in A\setminus\Omega_1(A)} [a,h]\right| \\
&= \sum_{h\in\Omega_1(A)\setminus\{0\}} \left|\bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ 2a=h}} [a,h]\right| \\
&\quad + \sum_{h\in\Omega_1(A)\setminus\{0\}} \left|\bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ 2a\neq h}} [a,h]\right| && \text{(by (33))} \\
&= \sum_{h\in\Omega_1(A)\setminus\{0\}} \frac{v}{2}|A \setminus \Omega_1(A)| && \text{(by (50), (51))}
\end{aligned}$$

$$= \frac{1}{2}v(v-\omega_1)(\omega_1-1). \tag{52}$$

If $H \in \left[{\Omega_1(A) \atop 2}\right]$ and $H = \langle h', h'' \rangle$, then

$$\left| \bigcup_{\{a,h\} \in \binom{H\setminus\{0\}}{2}} [a,h] \right| = |[h', h'']| = v \tag{53}$$

by (32) and (43). Thus we have

$$\begin{aligned}
\left| \bigcup_{h\in\Omega_1(A)\setminus\{0\}} \bigcup_{a\in\Omega_1(A)\setminus\langle h\rangle} [a,h] \right| &= \left| \bigcup_{H\in\left[{\Omega_1(A) \atop 2}\right]} \bigcup_{\{a,h\}\in\binom{H\setminus\{0\}}{2}} [a,h] \right| \\
&= \sum_{H\in\left[{\Omega_1(A) \atop 2}\right]} \left| \bigcup_{\{a,h\}\in\binom{H\setminus\{0\}}{2}} [a,h] \right| && \text{(by (32))} \\
&= \sum_{H\in\left[{\Omega_1(A) \atop 2}\right]} v && \text{(by (53))} \\
&= v \left| \begin{bmatrix} \Omega_1(A) \\ 2 \end{bmatrix} \right| \\
&= \frac{1}{6}v(\omega_1-1)(\omega_1-2) && \text{(by (23)).}
\end{aligned} \tag{54}$$

Therefore

$$\begin{aligned}
&|\{T \mid \mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_2\}| \\
&= \left| \bigcup_{\substack{T\in\binom{A}{3} \\ \mathrm{Orb}_{\hat{A}}(T)\in\mathcal{T}_2}} \mathrm{Orb}_{\hat{A}}(T) \right| \\
&= \left| \bigcup_{h\in\Omega_1(A)\setminus\{0\}} \bigcup_{a\in A\setminus\langle h\rangle} [a,h] \right| \\
&= \left| \bigcup_{h\in\Omega_1(A)\setminus\{0\}} \bigcup_{a\in A\setminus\Omega_1(A)} [a,h] \right| \\
&\quad + \left| \bigcup_{h\in\Omega_1(A)\setminus\{0\}} \bigcup_{a\in\Omega_1(A)\setminus\langle h\rangle} [a,h] \right| && \text{(by Lemma 4.4)} \\
&= \frac{1}{2}v(v-\omega_1)(\omega_1-1) + \frac{1}{6}v(\omega_1-1)(\omega_1-2) && \text{(by (52), (54))}
\end{aligned}$$

$$= \frac{1}{6}v(\omega_1 - 1)(3v - 2\omega_1 - 2). \tag{55}$$

On the other hand, counting the number of pairs (a, T) with $a \in A \setminus \Omega_2(A)$ and $T \in [a, -a]$ using (34), we find

$$2\left|\bigcup_{a\in A\setminus\Omega_2(A)} [a, -a]\right| = \sum_{a\in A\setminus\Omega_2(A)} |[a, -a]|. \tag{56}$$

Thus we have

$$\begin{aligned}
&|\{T \mid \mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1 \setminus \mathcal{T}_2\}| \\
&= \left|\bigcup_{\substack{T\in\binom{A}{3} \\ \mathrm{Orb}_{\hat{A}}(T)\in\mathcal{T}_1\setminus\mathcal{T}_2}} \mathrm{Orb}_{\hat{A}}(T)\right| \\
&= \left|\bigcup_{\substack{a\in A\setminus\Omega_1(A) \\ [a,-a]\notin\mathcal{T}_2}} [a, -a]\right| \\
&= \left|\bigcup_{a\in A\setminus\Omega_2(A)} [a, -a]\right| && \text{(by Lemma 4.3(i))} \\
&= \frac{1}{2}\sum_{a\in A\setminus\Omega_2(A)} |[a, -a]| && \text{(by (56))} \\
&= \frac{1}{2}v|A \setminus \Omega_2(A)| && \text{(by (41))} \\
&= \frac{1}{2}v(v - \omega_2).
\end{aligned}$$

Combining this with (55), we find

$$\begin{aligned}
|\{T \mid \mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1 \cup \mathcal{T}_2\}| &= \frac{1}{6}v(\omega_1 - 1)(3v - 2\omega_1 - 2) + \frac{1}{2}v(v - \omega_2) \\
&= \frac{1}{2}v^2\omega_1 - \frac{1}{6}v(2\omega_1^2 + 3\omega_2 - 2).
\end{aligned}$$

□

Next we compute the number of quadruples B satisfying $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{Q}_3$. Let

$$\begin{aligned}
\mathcal{B}_0 &= \{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{Q}_3\}, \\
A_0 &= \{a \in A \mid 2a = h_0\}, \\
\omega_0 &= |A_0|.
\end{aligned} \tag{57}$$

Lemma 4.9. *We have*

$$|\mathcal{B}_0| = \frac{1}{8}v^2\omega_1 - \frac{1}{24}v(2\omega_1^2 + 3\omega_2 - 2).$$

Proof. Counting the number of pairs (a, B) with $a \in A \setminus \Omega_1(A)$, $2a = h_0$ and $B \in [a, -a, h_0]$ using (35), we find

$$2\left|\bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ 2a=h_0}} [a, -a, h_0]\right| = \sum_{\substack{a\in A\setminus\Omega_1(A)\\ 2a=h_0}} |[a, -a, h_0]|.$$

Thus, by (42), we have

$$\begin{aligned}\left|\bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ 2a=h_0}} [a, -a, h_0]\right| &= \frac{v}{8}|\{a \in A \setminus \Omega_1(A) \mid 2a = h_0\}| \\ &= \frac{1}{8}v\omega_0. \qquad (58)\end{aligned}$$

Counting the number of pairs (a, B) with $a \in A\setminus\Omega_1(A)$, $2a \neq h_0$ and $B \in [a, -a, h_0]$ using (36), we find

$$4\left|\bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ 2a\neq h_0}} [a, -a, h_0]\right| = \sum_{\substack{a\in A\setminus\Omega_1(A)\\ 2a\neq h_0}} |[a, -a, h_0]|.$$

Thus, by (42), we have

$$\begin{aligned}\left|\bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ 2a\neq h_0}} [a, -a, h_0]\right| &= \frac{v}{4}|\{a \in A \setminus \Omega_1(A) \mid 2a \neq h_0\}| \\ &= \frac{1}{4}v(v - \omega_1 - \omega_0). \qquad (59)\end{aligned}$$

Therefore

$$\begin{aligned}&|\{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}_1\}| \\ &= \left|\bigcup_{a\in A\setminus\Omega_1(A)} [a, -a, h_0]\right|\end{aligned}$$

$$
\begin{aligned}
&= \left| \bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ 2a=h_0}} [a,-a,h_0] \right| \\
&\quad + \left| \bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ 2a\neq h_0}} [a,-a,h_0] \right| && \text{(by Lemma 4.2(ii))} \\
&= \frac{1}{8}vw_0 + \frac{1}{4}v(v-\omega_1-\omega_0) && \text{(by (58) and (59))} \\
&= \frac{1}{8}v(2v-2\omega_1-\omega_0). && (60)
\end{aligned}
$$

Let $h \in \Omega_1(A) \setminus \langle h_0\rangle$. Counting the number of pairs (a, B) with $a \in A \setminus \Omega_1(A)$, $2a \neq h$ and $B \in [a, h, h + a]$ using (40), we find

$$
4\left| \bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ 2a\neq h}} [a,h,h+a] \right| = \sum_{\substack{a\in A\setminus\Omega_1(A)\\ 2a\neq h}} |[a,h,h+a]|.
$$

Thus, by (44), we have

$$
\left| \bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ 2a\neq h}} [a,h,h+a] \right| = \frac{v}{8}|\{a\in A\setminus\Omega_1(A) \mid 2a\neq h\}|. \tag{61}
$$

Therefore

$$
\begin{aligned}
&|\{B \mid \mathrm{Orb}_{\hat{A}}(B)\in\mathcal{Q}_2\}| \\
&= \left| \bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ h\in\Omega_1(A)\setminus\langle h_0\rangle\\ 2a\neq h}} [a,h,a+h] \right| \\
&= \sum_{h\in\Omega_1(A)\setminus\langle h_0\rangle} \left| \bigcup_{\substack{a\in A\setminus\Omega_1(A)\\ 2a\neq h}} [a,h,a+h] \right| && \text{(by (33))} \\
&= \frac{v}{8} \sum_{h\in\Omega_1(A)\setminus\langle h_0\rangle} |\{a\in A\setminus\Omega_1(A) \mid 2a\neq h\}| && \text{(by (61))} \\
&= \frac{v}{8} \sum_{a\in A\setminus\Omega_1(A)} |(\Omega_1(A)\setminus\langle h_0\rangle)\setminus\{2a\}|
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{v}{8}\left(\sum_{a\in(A\setminus\Omega_2(A))\cup A_0} (\omega_1 - 2) + \sum_{a\in\Omega_2(A)\setminus(A_0\cup\Omega_1(A))} (\omega_1 - 3) \right) \\
&= \frac{v}{8}\left((v-\omega_2+\omega_0)(\omega_1-2) + (\omega_2-\omega_0-\omega_1)(\omega_1-3)\right) \\
&= \frac{v}{8}((\omega_1-2)v - \omega_1^2 + \omega_0 + 3\omega_1 - \omega_2). \qquad (62)
\end{aligned}
$$

If $H \in \begin{bmatrix}\Omega_1(A)\\2\end{bmatrix}$ and $H = \langle h', h''\rangle$, then

$$
\left| \bigcup_{\{h_1,h_2\}\in\binom{H\setminus\{0\}}{2}} [h_1, h_2, h_1+h_2] \right| = |[h', h'', h'+h'']| = \frac{v}{4} \qquad (63)
$$

by (32) and Lemma 4.7(ii). Thus we have

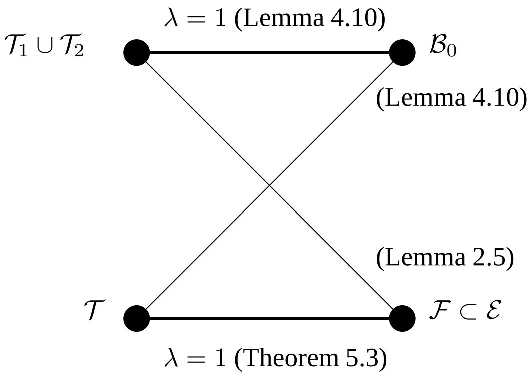
$$
\begin{aligned}
&|\{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}_3\}| \\
&= \left| \bigcup_{\substack{h,h'\in\Omega_1(A)\setminus\{0\}\\ h\neq h'}} [h, h', h+h'] \right| \\
&= \left| \bigcup_{H\in\begin{bmatrix}\Omega_1(A)\\2\end{bmatrix}} \bigcup_{\{h,h'\}\in\binom{H\setminus\{0\}}{2}} [h, h', h+h'] \right| \\
&= \sum_{H\in\begin{bmatrix}\Omega_1(A)\\2\end{bmatrix}} \left| \bigcup_{\{h,h'\}\in\binom{H\setminus\{0\}}{2}} [h, h', h+h'] \right| && \text{(by (32))} \\
&= \sum_{H\in\begin{bmatrix}\Omega_1(A)\\2\end{bmatrix}} \frac{v}{4} && \text{(by (63))} \\
&= \frac{v}{4}\left| \begin{bmatrix}\Omega_1(A)\\2\end{bmatrix} \right| \\
&= \frac{1}{24}v(\omega_1-1)(\omega_1-2) && \text{(by (23)).} \qquad (64)
\end{aligned}
$$

Since the sets $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are pairwise disjoint by Lemma 4.3(ii) and Lemma 4.4, the equations (60), (62) and (64) can be combined to give

$$
\begin{aligned}
|\mathcal{B}_0| &= |\{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}_1\}| + |\{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}_2\}| \\
&\quad + |\{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}_3\}| \\
&= \frac{1}{8}v(2v - 2\omega_1 - \omega_0) + \frac{1}{8}v((\omega_1-2)v - \omega_1^2 - \omega_2 + 3\omega_1 + \omega_0) \\
&\quad + \frac{1}{24}v(\omega_1-1)(\omega_1-2)
\end{aligned}
$$

$$= \frac{1}{8}v^2\omega_1 - \frac{1}{24}v(2\omega_1^2 + 3\omega_2 - 2).$$

$\square$



Lemma 4.10. *If $B \in \mathcal{B}_0$, $T \subset B$ and $|T| = 3$, then $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1 \cup \mathcal{T}_2$. Conversely, if $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1 \cup \mathcal{T}_2$, then there exists a unique $B \in \mathcal{B}_0$ such that $T \subset B$.*

Proof. Suppose $B \in \mathcal{B}_0$.

If $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}_1$, then we may assume without loss of generality that $B = \{0, a, -a, h_0\}$ for some $a \in A \setminus \Omega_1(A)$. If $T \in \{\{0, a, -a\}, \{a, -a, h_0\} = \{0, a + h_0, -(a + h_0)\} + h_0\}$, then $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1$. If $T = \pm\{0, a, h_0\}$, then $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_2$.

If $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}_2$, then we may assume without loss of generality that $B = \{0, a, h, h + a\}$ for some $a \in A \setminus \Omega_1(A)$ and $h \in \Omega_1(A) \setminus \langle h_0 \rangle$ with $2a \neq h$. Then for $T \in \{\{0, a, h\}, \{0, h + a, h\}, \{0, a, h + a\} = \{0, -a, h\} + a, \{a, h, h + a\} = \{0, h - a, h\} + a\}$, we have $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_2$.

If $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}_3$, then we may assume without loss of generality that $B = \{0, h, h', h+h'\}$ for some $h, h' \in \Omega_1(A)\setminus\{0\}$ with $h \neq h'$. Then for $T \in \{\{0, h, h'\}, \{0, h, h+h'\}, \{0, h', h + h'\}, \{h, h', h + h'\} = \{0, h, h'\} + h + h'\}$, we have $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_2$.

Conversely, suppose $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1 \cup \mathcal{T}_2$.

If $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1$, then we may assume without loss of generality that $T = \{0, a, -a\}$ for some $a \in A \setminus \Omega_1(A)$. Then $T \subset B = \{0, a, -a, h_0\}$ and $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}_1$, so $B \in \mathcal{B}_0$.

If $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_2$, then we may assume without loss of generality that $T = \{0, a, h\}$ for some $a \in A$ and $h \in \Omega_1(A) \setminus \{0, a\}$. If $a \in \Omega_1(A)$, then $T \subset B = \{0, a, h, h + a\}$ and $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}_3$, so $B \in \mathcal{B}_0$. If $a \notin \Omega_1(A)$ and $h = h_0$, then $T \subset B = \{0, a, -a, h_0\}$ and $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}_1$, so $B \in \mathcal{B}_0$. Suppose $a \notin \Omega_1(A)$ and $h \neq h_0$. If $2a \neq h$, then $T \subset B = \{0, a, h, h + a\}$ and $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}_2$, so $B \in \mathcal{B}_0$. If $2a = h$, then $T \subset B = \{0, a, h, h_0 + a\} = \{0, a, -a, h_0\} + a$ and $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}_1$, so $B \in \mathcal{B}_0$.

Therefore, we have shown that there exists at least one $B \in \mathcal{B}_0$ such that $T \subset B$. Counting the number of pairs (T, B) with $T \in \binom{A}{3}$, $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1 \cup \mathcal{T}_2$ and $T \subset B \in$

$\mathcal{B}_0$ using the first part, we find

$$\begin{aligned}4|\mathcal{B}_0| &= \sum_{\substack{T\in\binom{A}{3}\\ \mathrm{Orb}_{\hat{A}}(T)\in\mathcal{T}_1\cup\mathcal{T}_2}} |\{B\in\mathcal{B}_0 \mid T\subset B\}| \\ &\geq |\{T \mid \mathrm{Orb}_{\hat{A}}(T)\in\mathcal{T}_1\cup\mathcal{T}_2\}|.\end{aligned}$$

By Lemma 4.8 and Lemma 4.9, we conclude $|\{B \in \mathcal{B}_0 \mid T \subset B\}| = 1$ whenever $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1 \cup \mathcal{T}_2$. □

5 Reversible Steiner quadruple systems

In this section, we let A be an abelian group of order $v \equiv 2$ or $4 \pmod 6$, and use the notation introduced in Section 2–4. Specifically, we continue to use notation introduced in (3)–(4), Definition 2.1, (24)–(25), (28)–(30), and (57).

A quadruple $B \in \binom{A}{4}$ is said to be *symmetric* if $\mathrm{Orb}_{\hat{A}}(B) = \mathrm{Orb}_A(B)$.

Lemma 5.1. *A quadruple $B \in \binom{A}{4}$ is symmetric if and only if $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}' \cup \mathcal{Q}'' \cup \mathcal{Q}'''$, where*

$$\mathcal{Q}' = \{[a, b, a+b] \mid \{0, a, b, a+b\} \in \binom{A}{4}\}, \tag{65}$$

$$\mathcal{Q}'' = \{[a, -a, h] \mid \{0, a, -a, h\} \in \binom{A}{4},\ 2h = 0\}, \tag{66}$$

$$\mathcal{Q}''' = \{[h, h', h''] \mid \{h, h', h''\} \in \binom{\Omega_1(A)\setminus\{0\}}{3}\}. \tag{67}$$

In particular, every member of $\mathcal{B}_0$ is symmetric. Moreover, $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{E}$ implies that B is symmetric.

Proof. Suppose $B = \{0, a, b, a+b\} \in \binom{A}{4}$. Then $B^\sigma = B - (a+b) \in \mathrm{Orb}_A(B)$. Thus $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}'$ implies that B is symmetric.

Suppose $B = \{0, a, -a, h\} \in \binom{A}{4}$, $2h = 0$. Then $B^\sigma = B \in \mathrm{Orb}_A(B)$. Thus $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}''$ implies that B is symmetric.

Suppose $B = \{0, h, h', h''\} \in \binom{A}{4}$, $\{h, h', h''\} \in \binom{\Omega_1(A)\setminus\{0\}}{3}$. Then $B^\sigma = B \in \mathrm{Orb}_A(B)$. Thus $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}'''$ implies that B is symmetric.

Conversely, suppose that $B = \{0, a, b, c\} \in \binom{A}{4}$ is symmetric. Then

$$\begin{aligned}\{-a, -b, -c\} \in &\{\{a, b, c\}, \{-a, b-a, c-a\}, \\ &\{-b, a-b, c-b\}, \{-c, a-c, b-c\}\},\end{aligned}$$

hence

$$\begin{aligned}&\{-a, -b, -c\} = \{a, b, c\} \text{ or} \\ &\{-b, -c\} = \{b-a, c-a\} \text{ or} \\ &\{-a, -c\} = \{a-b, c-b\} \text{ or} \\ &\{-a, -b\} = \{a-c, b-c\}.\end{aligned}$$

Observe

$$\begin{aligned}
&\{-a,-b,-c\} = \{a,b,c\} \\
&\iff 2a = 0 \text{ and } \{b,c\} = \{-b,-c\} \text{ or} \\
&\qquad -a = b \text{ and } \{a,c\} = \{-b,-c\} \text{ or} \\
&\qquad -a = c \text{ and } \{a,b\} = \{-b,-c\} \text{ or} \\
&\iff 2a = 2b = 2c = 0 \text{ or } (2a,b) = (0,-c) \text{ or} \\
&\qquad -a = b \text{ and } 2c = 0 \text{ or} \\
&\qquad -a = c \text{ and } 2b = 0 \\
&\iff 2a = 2b = 2c = 0 \text{ or} \\
&\qquad (2a,b) = (0,-c) \text{ or} \\
&\qquad (2b,c) = (0,-a) \text{ or} \\
&\qquad (2c,a) = (0,-b) \\
&\implies \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}'' \cup \mathcal{Q}'''.
\end{aligned}$$

Also,

$$\begin{aligned}
&\{-b,-c\} = \{b-a, c-a\} \\
&\iff (-b,-c) = (b-a, c-a) \text{ or} \\
&\qquad (-b,-c) = (c-a, b-a) \\
&\iff B = \{0,b,-b,c-b\} + b \text{ and } 2(c-b) = 0 \text{ or} \\
&\qquad a = b + c \\
&\implies \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}' \cup \mathcal{Q}''.
\end{aligned}$$

Similarly, $\{-a,-c\} = \{a-b, c-b\}$ or $\{-a,-b\} = \{a-c, b-c\}$ implies $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}' \cup \mathcal{Q}''$.

It follows from the definitions (28), (29), (30), (65), (66) that $\mathcal{Q}_1 \subset \mathcal{Q}''$, $\mathcal{Q}_2 \cup \mathcal{Q}_3 \subset \mathcal{Q}'$. Thus by (57), we have $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}' \cup \mathcal{Q}''$ for any $B \in \mathcal{B}_0$. This implies that every member of $\mathcal{B}_0$ is symmetric.

Finally, as $\mathcal{E} \subset \mathcal{Q}'$, $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{E}$ implies that B is symmetric. $\square$

We remark that, if $v \equiv 2 \pmod 4$, then it is shown in Munemasa and Sawa (2007) that the set of all symmetric blocks forms an A-invariant 3-$(v,4,3)$ design.

Lemma 5.2. *If* $(A,\mathcal{B})$ *is an* A*-reversible* $\mathrm{SQS}(v)$ *such that* $\mathcal{B}_0 \subset \mathcal{B}$*, then* $\mathcal{B} \setminus \mathcal{B}_0 \subset \{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{E}\}$.

Proof. Let $B \in \mathcal{B}$. Since B is symmetric, $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}' \cup \mathcal{Q}'' \cup \mathcal{Q}'''$ by Lemma 5.1.

First suppose $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}''$. Then we may assume without loss of generality $B = \{0,a,-a,h\}$ for some $a \in A$ and $h \in \Omega_1(A) \setminus \{0\}$. Since $\{0,a,-a\} \subset \{0,a,-a,h_0\} \in \mathcal{B}_0 \subset \mathcal{B}$, we obtain $B = \{0,a,-a,h_0\} \in \mathcal{B}_0$.

Next suppose $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}'''$. Then we may assume without loss of generality $B = \{0,h,h',h''\}$ for some $h,h',h'' \in \Omega_1(A) \setminus \{0\}$. Since $\{0,h,h'\} \subset \{0,h,h',h+h'\} \in \mathcal{B}_0 \subset \mathcal{B}$, we obtain $B = \{0,h,h',h+h'\} \in \mathcal{B}_0$.

Finally, suppose $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q}'$. It suffices to show that $\mathrm{Orb}_{\hat{A}}(B) \notin \mathcal{E}$ implies $B \in \mathcal{B}_0$. We may assume without loss of generality $B = \{0, a, b, a+b\}$ for some $a, b \in A$. Since $\mathrm{Orb}_{\hat{A}}(B) \notin \mathcal{E}$, Lemma 2.2(ii) implies $0 \in \{2a, 2b\}$ or $\{\pm a, \pm 2a\} \cap \{\pm b, \pm 2b\} \neq \emptyset$. If the former occurs, then B contains a triple T with $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_2$. If the latter occurs with $0 \notin \{2a, 2b\}$, then we may assume $2a \in \{b, 2b\}$ by replacing a by $\pm b$ if necessary. If $2a = b$, then $B - a$ contains $T = \{0, a, -a\}$, and $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1$. If $2a = 2b$, then $B - a$ contains the triple $T = \{0, -a, b-a\}$, and we have $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_2$ since $2(b-a) = 0$. Therefore, we have shown that there exists $T \subset B$ such that $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1 \cup \mathcal{T}_2$. Note that the existence of an $\mathrm{SQS}(v)$ implies $v \equiv 2$ or $4 \pmod 6$, so we can apply Lemma 4.10 to conclude that there exists $B' \in \mathcal{B}_0 \subset \mathcal{B}$ such that $T \subset B'$. This forces $B = B' \in \mathcal{B}_0$. $\square$

Theorem 5.3. *Let A be an abelian group of order $v \equiv 2$ or $4 \pmod 6$. For a subset $\mathcal{B}$ of $\binom{A}{4}$ containing $\mathcal{B}_0$, the incidence structure $(A, \mathcal{B})$ is an A-reversible $\mathrm{SQS}(v)$ if and only if*

$$\mathcal{B} = \mathcal{B}_0 \cup \{B \in \binom{A}{4} \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{F}\}$$

for some 1-factor $\mathcal{F}$ of the Köhler graph of A.

Proof. By Lemma 4.10, we have

$$|\{B \in \mathcal{B}_0 \mid T \subset B\}| = 1 \text{ for } \forall T \in \binom{A}{3} \text{ with } \mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1 \cup \mathcal{T}_2, \tag{68}$$

and also, together with Lemma 4.1, we have

$$|\{B \in \mathcal{B}_0 \mid T \subset B\}| = 0 \text{ for } \forall T \in \binom{A}{3} \text{ with } \mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}. \tag{69}$$

Thus

$$\begin{aligned}
&(A, \mathcal{B}) \text{ is an } A\text{-reversible } \mathrm{SQS}(v) \\
\iff\ & (A, \mathcal{B}) \text{ is an } A\text{-reversible } \mathrm{SQS}(v), \\
& \mathcal{B} \setminus \mathcal{B}_0 \subset \{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{E}\} && \text{(by Lemma 5.2)} \\
\iff\ & \mathcal{B} \text{ is } \hat{A}\text{-invariant}, \\
& \text{every member of } \mathcal{B} \text{ is symmetric}, \\
& |\{B \in \mathcal{B} \mid T \subset B\}| = 1 \text{ for } \forall T \in \binom{A}{3}, \\
& \mathcal{B} \setminus \mathcal{B}_0 \subset \{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{E}\} \\
\iff\ & \mathcal{B} \text{ is } \hat{A}\text{-invariant}, \\
& |\{B \in \mathcal{B} \mid T \subset B\}| = 1 \text{ for } \forall T \in \binom{A}{3}, \\
& \mathcal{B} \setminus \mathcal{B}_0 \subset \{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{E}\} && \text{(by Lemma 5.1)}
\end{aligned}$$

$$
\begin{aligned}
\iff\ & \mathcal{B} \setminus \mathcal{B}_0 \text{ is } \hat{A}\text{-invariant},\\
& \mathcal{B} \setminus \mathcal{B}_0 \subset \{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{E}\},\\
& |\{B \in \mathcal{B} \mid T \subset B\}| = 1 \text{ for } \forall T \in \binom{A}{3}\\
\iff\ & \mathcal{B} \setminus \mathcal{B}_0 = \{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{F}\} \text{ for some } \mathcal{F} \subset \mathcal{E},\\
& |\{B \in \mathcal{B} \mid T \subset B\}| = 1 \text{ for } \forall T \in \binom{A}{3}\\
\iff\ & \mathcal{B} \setminus \mathcal{B}_0 = \{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{F}\} \text{ for some } \mathcal{F} \subset \mathcal{E},\\
& |\{B \in \mathcal{B} \setminus \mathcal{B}_0 \mid T \subset B\}| = 1\\
& \text{ for } \forall T \in \binom{A}{3} \text{ with } \mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T} \qquad\qquad \text{(by (68),(69))}\\
\iff\ & \mathcal{B} \setminus \mathcal{B}_0 = \{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{F}\} \text{ for some } \mathcal{F} \subset \mathcal{E},\\
& |\{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{F},\ T \subset B\}| = 1\\
& \text{ for } \forall T \in \binom{A}{3} \text{ with } \mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}.
\end{aligned}
$$

Now, for $T \in \binom{A}{3}$ with $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}$,

$$
\begin{aligned}
& |\{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{F},\ T \subset B\}|\\
&= |\{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{F},\ \mathrm{Orb}_{\hat{A}}(T) \text{ is incident with } \mathrm{Orb}_{\hat{A}}(B)\}|\\
&= |\{\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{F} \mid \mathrm{Orb}_{\hat{A}}(T) \text{ is incident with } \mathrm{Orb}_{\hat{A}}(B)\}|
\end{aligned}
$$

by Lemma 2.3. Therefore,

$$
\begin{aligned}
& (A, \mathcal{B}) \text{ is an } A\text{-reversible } \mathrm{SQS}(v)\\
& \iff \mathcal{B} \setminus \mathcal{B}_0 = \{B \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{F}\} \text{ for some 1-factor } \mathcal{F} \subset \mathcal{E}.
\end{aligned}
$$

□

Remark 5.4. We note that not all A-reversible $\mathrm{SQS}(v)$ are constructed using Theorem 5.3. Let $A = \langle h_0 \rangle \oplus \langle h \rangle \oplus \langle c \rangle$, where both h_0 and h have order 2, and c has order 5. Then, an A-reversible $\mathrm{SQS}(20)$ can be constructed by the quadruples given in Table 1. We note that for the block $B = \{0, c, -c + h, h\}$, we have $B \notin \mathcal{B}_0$ and $\mathrm{Orb}_{\hat{A}}(B) \notin \mathcal{E}$.

Theorem 5.5. *Let $n \geq 3$ be an integer, and let A be an abelian group of order 2^n whose exponent is 2 or 4. Then there exists an A-reversible* $\mathrm{SQS}(2^n)$.

Proof. By Theorem 5.3, it suffices to show that the Köhler graph of A has a 1-factor. By Lemma 3.8, it suffices to show that the Köhler graph of $\mathbb{Z}_2^{\epsilon_1} \oplus \mathbb{Z}_4^{\epsilon_2}$ has a 1-factor whenever $\epsilon_1 + \epsilon_2 \leq 2$. The Köhler graph of $\mathbb{Z}_2^{\epsilon_1} \oplus \mathbb{Z}_4^{\epsilon_2}$, where $\epsilon_1 + \epsilon_2 \leq 2$ is empty unless $(\epsilon_1, \epsilon_2) = (0, 2)$. By Example 3.4, the Köhler graph of $\mathbb{Z}_4^2$ is a 3-cube and hence has a 1-factor. □

$[h_0, h, h_0 + h]$
$[c, -c, h_0]$
$[2c, -2c, h_0]$
$[c + h, -c + h, h_0]$
$[2c + h, -2c + h, h_0]$
$[h_0 + h, c + h_0, c + h]$
$[h_0 + h, c, c + h_0 + h]$
$[h_0 + h, 2c, 2c + h_0 + h]$
$[h_0 + h, 2c + h_0, 2c + h]$
$[h, c + h_0, c + h_0 + h]$
$[h, 2c + h_0, 2c + h_0 + h]$
$[c + h_0, -2c + h_0, -c]$
$[c + h_0 + h, -2c + h_0 + h, -c]$
$[2c + h_0, c + h_0 + h, -2c + h]$
$[2c + h_0 + h, c + h, -2c + h_0]$
$[c, -c + h, h]$
$[2c, -2c + h, h]$

Table 1: SQS(20) with $\mathcal{B}_0 \not\subset \mathcal{B}$

6 Abelian groups with cyclic Sylow 2-subgroup

As in Section 2, we let A be an abelian group of order v, and use the notation introduced in Section 2. Moreover, in this section, we assume that the Sylow 2-subgroup of A is cyclic.

Lemma 6.1. *Suppose $a, b \in A$, and $\langle a, b\rangle$ is not cyclic. Then (21) holds, and $[a, b]$ is a vertex of degree 3 in the Köhler graph of A.*

Proof. Suppose contrary, that $2a \in \langle b\rangle$. Let S denote the Sylow 2-subgroup of A. Since $a + \langle b\rangle$ has order 2 in $A/\langle b\rangle$, it belongs to the Sylow 2-subgroup $(S + \langle b\rangle)/\langle b\rangle$ of $A/\langle b\rangle$. Hence we have $a \in S + \langle b\rangle$. Since S is the Sylow 2-subgroup which is cyclic, $S + \langle b\rangle$ is also cyclic. This implies that $\langle a, b\rangle$ is cyclic, contradicting the assumption. This proves (21).

It follows from (3) that $[a, b]$ is a vertex of $\mathcal{G}$. Also, it follows from Lemma 3.3 that $[a, b]$ has degree 3. $\square$

Remark 6.2. In general, there may be vertices of degree less than 3 in the Köhler graph of an abelian group A, if the Sylow 2-subgroup is not cyclic. Indeed, let $A = \langle x\rangle \oplus \langle y\rangle = \mathbb{Z}_2 \oplus \mathbb{Z}_{2m}$ for some $m \geq 3$. Let $a = x + y$, $b = -y$. Then $a \neq \pm b$, $2a = 2y \notin \{0, -y, -2y\} = \{0, b, 2b\}$, and $2b = -2y \notin \{0, x + y, 2y\} = \{0, a, 2a\}$. Thus $[a, b] \in \mathcal{T}$ and $2a + 2b = 0$. It follows from Lemmas 3.2 and 3.3 that the vertex $[a, b]$ of the Köhler graph of A has degree less than 3.

Lemma 6.3. *Suppose $a, b \in A$, and $\langle a, b\rangle$ is not cyclic. Then the connected component containing $[a, b]$, of the Köhler graph of A is 3-regular and 2-edge-connected, and in particular has a 1-factor.*

Proof. First note that, by Lemma 6.1, $[a,b]$ is indeed a vertex of $\mathcal{G}$.

Suppose that $[c,d] \in \mathcal{T}$ is a vertex in the connected component $\mathcal{C}$ of the Köhler graph of A containing $[a,b]$. By Lemma 3.7, we have $\langle a,b\rangle = \langle c,d\rangle$, hence $[c,d]$ has degree 3 by Lemma 6.1. This proves that $\mathcal{C}$ is 3-regular.

To prove the 2-edge-connectivity, it suffices to find a cycle containing a given edge. Without loss of generality, we may assume that a given edge is incident with the vertex $[a,b]$. Then by Lemma 3.2, there are three possibilities:

$$[a,b,a+b], [a,b,a-b], [a,b,b-a].$$

In the latter two cases, we may replace (a,b) by $(a-b,b)$, $(a,b-a)$, respectively, to reduce to the first case. Since $\langle a,b\rangle$ is not cyclic, (21) holds by Lemma 6.1, and hence by Lemma 3.9, there exists a cycle containing $[a,b,a+b]$. We thus conclude that $\mathcal{C}$ is 2-edge-connected. It is well known in graph theory (Petersen (1891), see also Lovász (1993)[p.59]) that any 2-edge-connected and 3-regular graph has a 1-factor, which completes the proof. $\square$

Now let A be an abelian group of order $v \equiv 2$ or $4 \pmod 6$. We fix an element $h_0 \in A$ of order 2, and use the notation introduced in (24)–(25), (28)–(30), and (57).

Lemma 6.4. *Let A be an abelian group of order v whose Sylow 2-subgroup is cyclic. If $(A,\mathcal{B})$ is an A-reversible $\mathrm{SQS}(v)$, then $\mathcal{B}_0 \subset \mathcal{B}$.*

Proof. Observe $\mathcal{Q}_2 = \mathcal{Q}_3 = \emptyset$. Thus it suffices to show $\{0,a,-a,h_0\} \in \mathcal{B}$ for any $a \in A \setminus \Omega_1(A)$. Let $B \in \mathcal{B}$ be the unique block containing $\{0,a,-a\}$. Then $B = \{0,a,-a,b\}$ for some $b \in A$. Since $\{0,a,-a\} \subset B \cap (-B)$ and $-B \in \mathcal{B}$, we must have $B = -B$, which implies b has order 2. Since the Sylow 2-subgroup of A is cyclic, h_0 is the unique element of order 2, hence $b = h_0$. $\square$

Theorem 6.5. *Let A be an abelian group of order $v \equiv 2$ or $4 \pmod 6$ such that the Sylow 2-subgroup of A is cyclic. Then there exists an A-reversible $\mathrm{SQS}(v)$ if and only if the Köhler graph of A has a 1-factor.*

Proof. Immediate from Theorem 5.3 and Lemma 6.4. $\square$

7 Main theorem

The following theorem is an extension of a theorem of Piotrowski (1985).

Theorem 7.1. *Let $v \geq 8$ be a positive integer. The following statements are equivalent:*

(i) *There exists an A-reversible $\mathrm{SQS}(v)$ for any abelian group A of order v whose Sylow 2-subgroup is cyclic;*

(ii) *There exists an A-reversible $\mathrm{SQS}(v)$ for some abelian group A of order v whose Sylow 2-subgroup is cyclic;*

(iii) *There exists an S-cyclic $\mathrm{SQS}(v)$;*

(iv) $v \equiv 0 \pmod 2$, $v \not\equiv 0 \pmod 3$, $v \not\equiv 0 \pmod 8$, *and there exists an S-cyclic* $\mathrm{SQS}(2p)$ *for any odd prime divisor p of v.*

Proof. Clearly, (i) implies (ii) and (iii). The equivalence of (iii) and (iv) is due to Piotrowski (1985)[Satz 14.1]. So it remains to prove (ii) $\Longrightarrow$ (iv) and (iv) $\Longrightarrow$ (i).

Suppose (ii) holds. Let A be an abelian group of order v whose Sylow 2-subgroup is cyclic, and suppose that there exists an A-reversible $\mathrm{SQS}(v)$. This implies $v \equiv 0 \pmod 2$ and $v \not\equiv 0 \pmod 3$. By Theorem 6.5, the Köhler graph of A has a 1-factor. Since A has a cyclic subgroup of order $2p$ for any odd prime divisor p of v, Lemma 3.8 implies that the Köhler graph of a cyclic group of order $2p$ has a 1-factor. By Theorem 6.5 again, there exists a $\mathbb{Z}_{2p}$-reversible $\mathrm{SQS}(2p)$. Also, as the Köhler graph of a cyclic group of order 8 has no 1-factor, A has no element of order 8. Since the Sylow 2-subgroup of A is cyclic, this implies $v \not\equiv 0 \pmod 8$. Therefore, (iv) holds.

Next we prove (iv) $\Longrightarrow$ (i). Let A be an arbitrary abelian group of order v whose Sylow 2-subgroup is cyclic. In view of Theorem 6.5, it suffices to show that the Köhler graph of A has a 1-factor. Let $[a, b]$ be a vertex of the Köhler graph $\mathcal{G}$ of A. The connected component of $\mathcal{G}$ containing $[a, b]$ is isomorphic to a connected component of the Köhler graph of $A' = \langle a, b \rangle$, by Lemma 3.8.

If A' is not cyclic, then by Lemma 6.3, the connected component of $\mathcal{G}$ containing $[a, b]$ has a 1-factor.

Suppose that A' is cyclic. If A' has odd order, then there exists a cyclic subgroup $\tilde{A}$ containing A' with $|\tilde{A} : A'| = 2$. By the implication (iv) $\Longrightarrow$ (iii), there exists an S-cyclic $\mathrm{SQS}(2|A'|)$. It follows from Theorem 6.5 that there exists a 1-factor in the Köhler graph of $\tilde{A}$. By Lemma 3.8, there exists a 1-factor in the Köhler graph of A', and in particular, there exists a 1-factor in the connected component of $\mathcal{G}$ containing $[a, b]$. □

In Siemon (1991), it is shown that a S-cyclic $\mathrm{SQS}(2p)$ exists for any prime number $p \equiv 53$ or $77 \pmod{120}$ with $p < 500000$. Applying Theorem 7.1 to this result shows that there exists an A-reversible $\mathrm{SQS}(v)$ for any abelian group A of order v which is twice a product of prime numbers p with $p \equiv 53$ or $77 \pmod{120}$ and $p < 500000$. The interested reader is also referred to Kaski et al. (2006); Huber (2010) for recent results on SQS with various automorphism groups.

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